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TICAL ANALYSIS OF VERY HIGH-DIMENSIONAL DATA
HICALLY STRUCTURED BINARY VARIABLES WITH MIS
PPLICATION TO MARINE CORPS READINESS EVALUAT

by

S. Zacks
W. H. Marlow
S. S. Brier

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STATISTICAL ANALYSIS OF VERY HIGH-DIMENSIONAL DATA SETS OF HIERARCHICALLY STRUCTURED BINARY VARIABLES WITH MISSING DATA AND APPLICATION TO MARINE CORPS READINESS EVALUATIONS

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READINESS RESEARCH
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31 December 1983

THE GEORGE WASHINGTON UNIVERSITY
School of Engineering and Applied Science
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1. Introduction

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The present analysis deals with very high-dimensional data sets, each one containing close to nine hundred binary variables. Each data set corresponds to an evaluation of one complex system. These data sets are characterized by large portions of missing data where, moreover, the unobserved variables are not the same in different evaluations. Thus, the problems which confront the statistical analysis are those of multivariate binary data analysis, where the number of variables is much larger than the sample size and in which missing data varies with the sample elements. The variables, however, are hierarchically structured and the problem of clustering variables to groups does not exist in the present study. In order to motivate the statistical problem under consideration, the Marine Corps Combat Readiness Evaluation System (MCCRES) is described for infantry battalions and then used for exposition. The present paper provides a statistical model for data from MCCRES and develops estimation and prediction procedures which utilize the dependence structure. The E-M algorithm is applied to obtain maximum-likelihood estimates of the parameters of interest. Numerical examples illustrate the proposed methods.

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The Marine Corps Combat Readiness Evaluation System measures the effectiveness of Marine Corps units, including infantry units (battalions), against well defined Mission Performance Standards (MPS) in simulated combat environments. The evaluation system for infantry battalions is comprised of 17 MPS's which are classified into four sections. Section A contains three MPS's which are applicable to all evaluations. Section B contains seven MPS's pertaining to amphibious assaults and normal combat operations. Section C has four MPS's on specialized combat operations and Section D has three which deal with use of outside support assets. In all cases, each MPS consists of several tasks and each task consists of several requirements.

In the present paper the term evaluation refers to the process of evaluating the readiness of an infantry battalion. In an evaluation a team of judges assigns the combat unit a value of 1 or 0 on each requirement being evaluated, according to whether the performance for that requirement is satisfactory or not and an N/A if the requirement was not applicable (was not evaluated). A weighted average of all the scores for the applicable requirements in a given task is then determined. For example, if in a given task there are 6 requirements, and the given data are

requirement	1	2	3	4	5	6
score	0	1	1	N/A	N/A	1
weight	.25	.083	.083	.083	.25	.25

The weighted applicable task score is

$$\frac{0 \times .25 + 1 \times .083 + 1 \times .083 + 1 \times .25}{.25 + .083 + .083 + .25} = .625$$

The weights for these requirements have been determined for the MCCRES by teams of experts. Accordingly, each applicable task is represented by a weighted average, ranging between zero and one. This weighted average will be called the observed task-score. The present MCCRES system computes hierarchically a weighted average of all the task-scores within an MPS, to obtain an MPS-score for each unit; a weighted average of all the MPS-scores to obtain a section-score and a weighted average of all section-scores to obtain a summarizing total-evaluations-score (TES). These weighted averages are computed as in the case of the observed task scores. If all the requirements in a given task are N/A, the task itself is called non-applicable (N/A). Similarly, if all tasks in a given MPS are N/A the MPS is called N/A. The weighted averages of the task scores in an MPS are based only on the applicable tasks. Similarly, the weighted averages of MPS scores in the four sections are based only on the applicable MPS scores. Thus, every unit can be characterized after an evaluation by a profile consisting of seventeen vectors of task-scores, one for each MPS; by four vectors of MPS-scores, one for each section; by a vector of four section-scores and a TES. In Table 1.1 we provide an example of such a profile of scores for evaluation No. 1. Other evaluations have the same hierarchical scores structure but the non-applicable tasks or MPS's might vary from one evaluation to another. This lack of uniformity in the applicable and non-applicable requirements from one evaluation to another creates severe data deficiency. The observed task scores may not reflect correctly the ability of a unit (battalion) to perform a given task. The task scores depend heavily on the design of the evaluation in terms of which requirements are non-applicable. This can be illustrated by using the previous example. If requirements 4 and 5 were applicable and the scores obtained on these requirements are 1 and 0, respectively, the task score would have been $0 \times .25 + 1 \times .083 + 1 \times .083 + 1 \times .083 + 0 \times .25 + 1 \times .25 = .50$.

Table 1.1

Hierarchical Scores of A Particular Evaluation*)

Level												
TES	.800											
Section	A	B	C	D								
	.745	.843	.863	.930								
MPS	1	2	3	4	5	6	7					
2.A.	.633	.928	.755									
2.B.	.935	.718	.854	.850	.XXX	.843	.849					
2.C.	.XXX	.772	.XXX	1.000								
2.D.	1.000	.841	.889									
TASK	1	2	3	4	5	6	7	8	9	10	11	12
2.A.1.	.757	.568	.848	.575	.500	.550	.331	.857	.593	.429	.833	.647
2.A.2.	.940	1.000	1.000	.855	1.000	1.000	.783	.857	.852	1.000	1.000	.937
2.A.3.	.742	.770	1.000	1.000	1.000	.667	.603	.239				
2.B.1.	.962	1.000	1.000	.733								
2.B.2.	1.000	1.000	1.000	.536	1.000	.500	.652					
2.B.3.	1.000	.960	1.000	.680	.865							
2.B.4.	.827	.680	.743	.625	1.000	.839	1.000	1.000	.XXX			
2.B.5.	.XXX	.XXX	.XXX	.XXX	.XXX	.XXX	.XXX					
2.B.6.	.797	.891	.800	.XXX	.XXX	.XXX						
2.B.7.	1.000	.727	.XXX	.XXX	.862	.XXX						
2.C.1.	.XXX	.XXX	.XXX	.XXX	.XXX							
2.C.2.	1.000	1.000	.543									
2.C.3.	.XXX	.XXX	.XXX	.XXX	.XXX	.XXX						
2.C.4.	1.000	1.000	1.000									
2.D.1.	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000			
2.D.2.	.851	1.000	.316	1.000								
2.D.3.	.888	.890										

*) .XXX represents a non-applicable task or a non-applicable MPS.

The previous score of .625 is overestimating the performance of the task by 25%. The problem is therefore to develop a method of assessing the statistical characteristics of the task scores in the population of all possible evaluations, and suggest a method of adjusting the individual task scores, in cases of N/A requirements, to better estimate the capability of the unit. The present paper addresses this problem of developing proper estimation procedures of the basic structural parameters, so that meaningful prediction of the missing values and adjustment of the task scores will be possible. The analysis is based on a data set of 75 evaluations. These evaluations were conducted on different combat units, some of which were evaluated only partially. In particular, Sections B, C and D of the evaluations have a large number of N/A requirements in many evaluations. We therefore restrict attention in the present paper to Section A, which consists of 3 MPS's encompassing 31 tasks. As mentioned earlier, the statistical problem discussed in the present paper is that of analyzing multivariate binary data sets having an extensive amount of incomplete or missing data. As such, the interest in the methods developed here is beyond the particular scope of MCCRES evaluations. The results of the present study may well be applicable to the fields of reliability of complex systems, signal processing, etc.

2. The Data Structure and Basic Definitions

In the present section we discuss the structure of the data file, on which the analysis of the present study is based, and provide some basic definitions. As mentioned earlier, we concentrate on Section 2.A. The discussion is therefore confined to the structure of the data in this section. The data related to the other three sections is structured similarly. As seen in Table 1.1, Section A is comprised of three MPS's, consisting of 12, 12 and 7 tasks, respectively. Each of these 31 tasks is structured around a certain number of requirements. The number of requirements may differ from one task to another. Moreover, not all the units were evaluated on the same requirements. Thus, if a unit was not evaluated on a given requirement, that requirement is non-applicable for that unit. In Table 2.1 we present data on requirements of two particular evaluations. The value 1 signifies that the requirement was performed satisfactorily. The value 0 was assigned if the performance was unsatisfactory and the value 9 designates a non-applicable requirement. We see in Table 2.1 that the number of requirements for the 31 tasks of Section 2.A are, respectively, 9, 5, 5, 5, 9, 10, 10, 8, 10, 7, 6, 6, 7, 4, 7, 8, 9, 8, 7, 10, 9, 8, 6, 5, 11, 7, 6, 7, 9, 9 and 7. We also see that in EVAL 1 only requirement 2.A.2.11.5 was not applicable while in EVAL 2 there are several such requirements. In addition, there are differences due to the performance level of the two units. We define below the pertinent variables for the structure of the data in the present study.

Let i denote the index of an MPS within Section 2.A, $i = 1, 2, 3$. Let $t_i (i = 1, 2, 3)$ denote the number of tasks in the i-th MPS of 2.A, and let j, $j = 1,\dots,t_i$, denote the index of the j-th task of MPS 2.A.i. Let $r_{ij} (i = 1,\dots,3;\ j = 1,\dots,t_i)$ denote the number of requirements in task 2.A.i.j and let k, $k = 1,\dots,r_{ij}$, be the index of the k-th requirement within 2.A.i.j. Finally, let n, $n = 1,\dots,N$ $(N = 75)$, be the index of the n-th evaluation (unit). We introduce two basic

Table 2.1

Requirement Data For Evaluations 1 and 2

EVAL 1

Task	Req.										
2.A.1.01	1	0	1	0	1	1	1	1	1		
2.A.1.02	0	1	1	1	0						
2.A.1.03	1	0	1	1	1						
2.A.1.04	0	0	1	1	1						
2.A.1.05	1	0	0	1	1	0	0	1	1		
2.A.1.06	1	0	1	0	1	0	1	0	1	1	
2.A.1.07	1	0	0	0	0	1	0	0	1	1	
2.A.1.08	1	1	0	1	1	1	1	1			
2.A.1.09	1	1	1	0	0	1	0	0	1	1	
2.A.1.10	1	1	1	0	0	0	0				
2.A.1.11	1	1	1	0	1	1					
2.A.1.12	1	1	0	0	1	1					
2.A.2.01	0	1	1	1	1	1	1				
2.A.2.02	1	1	1	1							
2.A.2.03	1	1	1	1	1	1	1				
2.A.2.04	1	1	1	0	1	1	1	1			
2.A.2.05	1	1	1	1	1	1	1	1	1		
2.A.2.06	1	1	1	1	1	1	1	1			
2.A.2.07	1	1	0	1	1	1	1				
2.A.2.08	1	1	1	1	0	1	1	1	1	1	
2.A.2.09	1	1	1	1	1	0	1	1	1		
2.A.2.10	1	1	1	1	1	1	1	1			
2.A.2.11	1	1	1	1	9	1					
2.A.2.12	0	1	1	1	1						
2.A.3.01	1	1	1	0	1	1	0	0	1	1	1
2.A.3.02	1	1	1	0	0	1	1				
2.A.3.03	1	1	1	1	1	1					
2.A.3.04	1	1	1	1	1	1	1				
2.A.3.05	1	1	0	1	1	0	1	1	0		
2.A.3.06	0	1	1	1	0	0	1	1	1		
2.A.3.07	0	1	1	0	0	0	0				

EVAL 2

Task	Req.										
2.A.1.01	1	0	1	0	1	0	1	1	0		
2.A.1.02	1	1	1	1	1						
2.A.1.03	1	1	1	0	1						
2.A.1.04	0	0	0	0	1						
2.A.1.05	1	1	1	1	1	1	0	1	1		
2.A.1.06	1	1	1	1	1	1	1	1	1	1	
2.A.1.07	1	1	1	1	1	1	1	1	1	1	
2.A.1.08	1	1	1	1	0	0	1	1			
2.A.1.09	0	0	1	1	0	0	1	1	1	1	
2.A.1.10	0	0	0	9	9	9	9				
2.A.1.11	1	1	1	1	1	1					
2.A.1.12	1	1	0	0	1	1					
2.A.2.01	1	1	1	1	1	1	1				
2.A.2.02	1	1	1	1							
2.A.2.03	1	1	1	1	1	1	1				
2.A.2.04	1	1	1	1	1	1	9	9			
2.A.2.05	1	1	1	1	1	1	1	1	1		
2.A.2.06	1	1	1	1	1	1	1	1			
2.A.2.07	1	1	1	1	1	1	1				
2.A.2.08	1	1	1	1	1	0	1	1	1	1	
2.A.2.09	1	1	1	1	1	1	1	1	1		
2.A.2.10	1	1	1	1	1	1	1	1			
2.A.2.11	0	0	1	9	1	1					
2.A.2.12	1	1	1	1	1						
2.A.3.01	1	1	1	1	1	1	1	1	1	1	1
2.A.3.02	1	1	1	1	1	1	1				
2.A.3.03	1	1	1	1	1	1					
2.A.3.04	1	1	1	0	1	1	1				
2.A.3.05	1	1	1	1	1	1	1	1	0		
2.A.3.06	1	1	1	0	0	0	0	1	0		
2.A.3.07	1	1	0	1	1	0	1				

variables: <u>an incidental variable</u>

(2.1) $$Z_{ijk}^{(n)} = \begin{cases} 1 \text{ , if requirement 2.A.i.j.k is applicable for the n-th evaluation} \\ 0 \text{ , otherwise} \end{cases}$$

and <u>a random variable</u>

(2.2) $$J_{ijk}^{(n)} = \begin{cases} 1 \text{ , if requirement 2.A.i.j.k is satisfied at the n-th evaluation} \\ 0 \text{ , otherwise} \end{cases}$$

The incidental variables $Z_{ijk}^{(n)}$ are design dependent, while the random variables $J_{ijk}^{(n)}$ follow some probabilistic model which is specified in Section 3.

Let w_{ijk} denote the weight assigned to requirement 2.A.i.j.k . These are non-negative values such that $\sum_{k=1}^{r_{ij}} w_{ijk} = 1$ for all i,j .

Let

(2.3) $$W_{ij}^{(n)} = \sum_{k=1}^{r_{ij}} Z_{ijk}^{(n)} w_{ijk}$$

be the sum of applicable weights for task 2.A.i.j at the n-th evaluations. Accordingly, the task-score $X_{ij}^{(n)}$ of the n-th evaluation is the random variable

(2.4) $$X_{ij}^{(n)} = \sum_{k=1}^{r_{ij}} Z_{ijk}^{(n)} J_{ijk}^{(n)} w_{ijk}/W_{ij}^{(n)} .$$

If at the n-th evaluation <u>all</u> the r_{ij} requirements of task 2.A.i.j are not applicable then $W_{ij}^{(n)} = 0$ and we define $X_{ij}^{(n)} = N/A$. If $Z_{ijk}^{(n)} = 1$ for all $k=1,\ldots,r_{ij}$ we say that the task score $X_{ij}^{(n)}$ is <u>complete</u>.

If some requirements are non-applicable the corresponding task score is incomplete.

3. The Statistical Model and the Basic Parameters

Let $\theta_{ijk}^{(n)}$ be the probability that requirement i.j.k is satisfied at the n-th evaluation. These values vary from one evaluation to another according to some distribution of these parameters. More specifically, we assume that,

$$P\left\{ J_{ijk}^{(n)} = 1 \mid \theta_{ijk}^{(n)} \right\} = \theta_{ijk}^{(n)} \tag{3.1}$$

we further assume that $\left\{ J_{ijk}^{(n)} \; ; \; n=1,2,\ldots,N \right\}$ are conditionally independent, given $\{\theta_{ijk}^{(n)} \; ; \; n=1,\ldots,N\}$ and that $\{\theta_{ijk}^{(n)} \; ; \; n=1,\ldots,N\}$ are independent random variables, having a common (unknown) distribution.

Let $\Pi_{ijk} = E\,\{\theta_{ijk}^{(n)}\}$. Thus, $P\,\{J_{ijk}^{(n)} = 1\} = \Pi_{ijk}$ for all $n=1,\ldots,N$. Furthermore, $\{J_{ijk}^{(n)} \; ; \; n=1,\ldots,N\}$ can be considered as a sequence of independent random variables representing N Bernoulli trials. Thus, Π_{ijk} is the probability that a randomly selected evaluation has satisfied requirement i.j.k. The expected complete task score (ECTS) for a randomly selected evaluation is

$$\tau_{ij} = \sum_{k=1}^{r_{ij}} w_{ijk} \, \Pi_{ijk} \tag{3.2}$$

Other parameters of interest of the (super) population of all possible evaluations are:

(i) Variances and covariances of $J_{ijk}^{(n)}$;

(ii) Variances and covariances of CTS's.

(iii) Correlations among CTS's.

The covariances of $J_{ijk}^{(n)}$ and $J_{ijk'}^{(n)}$, within a randomly chosen evaluation, are:

$$(3.3) \qquad \operatorname{cov}\left(J_{ijk}^{(n)} , J_{ijk'}^{(n)}\right) = \begin{cases} \Pi_{ijk} (1-\Pi_{ijk}) & , k = k' \\ P_{ij}(k,k') - \Pi_{ijk} \Pi_{ijk'}, & k \neq k' \end{cases}$$

where

$$(3.4) \qquad P_{ij}(k,k') = P \left\{ J_{ijk}^{(n)} \; J_{ijk'}^{(n)} = 1 \right\} .$$

The covariances given by (3.3) will be denoted by $\sigma_{ij}(k,k')$, and the covariance matrix of requirements within the i.j task is denoted by (V_{ij}) . (V_{ij}) is a positive definite matrix for every i.j . The variance of a complete task score, $\tilde{X}_{ij}^{(n)} = \sum_{k=1}^{r_{ij}} w_{ijk} J_{ijk}^{(n)}$, is given by

$$(3.5) \qquad \sigma_{ij}^2 = V \left\{ \tilde{X}_{ij}^{(n)} \right\} = \underset{\sim}{w}'_{ij} (V_{ij}) \underset{\sim}{w}_{ij} ,$$

where $\underset{\sim}{w}'_{ij} = (w_{ij1},\ldots,w_{ijr_{ij}})$ is the vector of weights of requirements within the (i.j)th task. The covariance of two different complete task scores, $\tilde{X}_{ij}^{(n)}$ and $\tilde{X}_{ij'}^{(n)}$, within the i-th MPS can be written as

$$(3.6) \qquad \operatorname{cov}(\tilde{X}_{ij}^{(n)} , \tilde{X}_{ij'}^{(n)}) = \underset{\sim}{w}'_{ij} (K_{i\cdot jj'}) \underset{\sim}{w}_{ij'}$$

where $(K_{i\cdot jj'})$ is an $r_{ij} \times r_{ij'}$ matrix of covariances $C_{i\cdot jj'}(k,\ell) = \operatorname{cov}(J_{ijk}^{(n)} , J_{ij'\ell}^{(n)})$.

Extending the previous definition, we have

$$(3.7) \qquad C_{i\cdot jj'}(k,\ell) = P_{i\cdot jj'}(k,\ell) - \Pi_{ijk} \Pi_{ij'\ell} ,$$

where $P_{i\cdot jj'}(k,\ell) = P\left\{J_{ijk}^{(n)}\ J_{ij'\ell}^{(n)} = 1\right\}$. Let $C_i(j,j')$ denote the covariances defined in (3.6). The correlations between two complete task scores in the same MPS are then

$$\rho_{i\cdot jj'} = C_i(j,j')/\ \sigma_{ij}\ \sigma_{ij'}\ ,\ j,j' = 1,\ldots,t_i\ . \tag{3.8}$$

In the following section we present estimators of these parameters and discuss their properties.

4. Estimating the Basic Parameters

Let $M_{ijk}^{(N)}$ denote the number of evaluations, among the N given ones, in which requirement i.j.k is applicable, i.e.,

$$M_{ijk}^{(N)} = \sum_{n=1}^{N} Z_{ijk}^{(n)}$$

We assume that

(i) $M_{ijk}^{(N)} > 0$, for all i.j.k ,

and

(ii) $M_{ijk}^{(N)} \to \infty$ as $N \to \infty$, for all i.j.k.

4.1 Estimating ECTS

The following is an unbiased and strongly consistent estimator of Π_{ijk} ; namely,

$$\hat{\Pi}_{ijk}^{(N)} = \sum_{n=1}^{N} Z_{ijk}^{(n)}\ J_{ijk}^{(n)}\ /\ M_{ijk}^{(N)} \tag{4.1}$$

Indeed, according to assumption (i), $E\left\{\hat{\Pi}_{ijk}^{(N)}\right\} = \Pi_{ijk}$ and, according to assumption (ii) and the strong law of large numbers (SLLN)

$\hat{\Pi}_{ijk}^{(N)} \xrightarrow{a.s.} \Pi_{ijk}$ as $N \to \infty$. Moreover, since for each i.j.k, $J_{ijk}^{(n)}$ are independent and identically distributed (IID), n=1,...,N, the distribution of $S_{ijk}^{(N)} = \sum_{n=1}^{N} Z_{ijk}^{(n)} J_{ijk}^{(n)}$ is the binomial, $B(M_{ijk}^{(N)}, \Pi_{ijk})$. Thus, the estimator $\hat{\Pi}_{ijk}^{(N)}$ is asymptotically normal with mean Π_{ijk} and variance $\sigma_{ij}(k,k)/M_{ijk}^{(N)}$, where $\sigma_{ij}(k,k) = \Pi_{ijk}(1-\Pi_{ijk})$. Thus, confidence intervals for Π_{ijk} can be determined in the usual manner (see Zacks [1981, pp. 268]).

An <u>unbiased</u> and <u>strongly consistent</u> estimator of the ECTS, τ_{ij}, is

$$\hat{\tau}_{ij}^{(N)} = \sum_{k=1}^{r_{ij}} w_{ijk} \hat{\Pi}_{ijk}^{(N)} \tag{4.2}$$

Indeed, $\hat{\tau}_{ij}^{(N)}$ is a linear combination of unbiased and strongly consistent estimators. The <u>variance</u> of $\hat{\tau}_{ij}^{(N)}$ is given by

$$V\left\{\hat{\tau}_{ij}^{(N)}\right\} = \underset{\sim}{w}_{ij}' \left(D_{ij}^{(N)}\right) \underset{\sim}{w}_{ij}, \tag{4.3}$$

where $(D_{ij}^{(N)})$ is a positive definite symmetric matrix whose elements are

$$D_{ij}^{(N)}(k,k') = \sigma_{ij}(k,k') \frac{M_{ij}^{(N)}(k,k')}{M_{ijk}^{(N)} M_{ijk'}^{(N)}} \quad ; \quad k,k'=1,\dots,r_{ij} ; \tag{4.4}$$

where

$$M_{ij}^{(N)}(k,k') = \sum_{n=1}^{N} Z_{ijk}^{(n)} Z_{ijk'}^{(n)} . \tag{4.5}$$

Notice that $D_{ij}^{(N)}(k,k')$ depends on the $Z_{ijk}^{(n)}$ values. We assume that

$M^{(N)}_{ijk}(k,k') > 0$ for all i.j and all $k,k'=1,\dots,r_{ij}$. Moreover, we assume

(iii) $M^{(N)}_{ij}(k,k') \to \infty$ as $N \to \infty$, all i.j, $k,k'=1,\dots,r_{ij}$.

If $Z^{(n)}_{ijk} = 1$ for all i.j.k and all $n=1,\dots,N$, then $D^{(N)}_{ij}(k,k') = \sigma_{ij}(k,k')/N$. This is generally not the case! It is simple to show that

$$\frac{M^{(N)}_{ij}(k,k')}{M^{(N)}_{ijk}\, M^{(N)}_{ijk'}} \le \frac{1}{\mathrm{Max}(M^{(N)}_{ijk}\,,\, M^{(N)}_{ijk'})} \quad , \begin{array}{l}\text{all } i.j\\ \text{all } k,k'\end{array} \tag{4.6}$$

Thus, if $M^{(N)}_{ijk} = O(N)$ then $V\left\{\hat{\tau}^{(N)}_{ijk}\right\} = O(N^{-1})$. In Section 5 we will develop estimators of Π_{ijk} and of τ_{ij} , which utilize the covariances between requirements.

4.2 Estimating Variances and Covariances

We start with the development of estimators of the variances and covariances of the requirement variables $J^{(n)}_{ijk}$, i.e. $\sigma_{ij}(k,k')$. A <u>strongly consistent estimator</u> of $\sigma_{ij}(k,k')$ is

$$\hat{\sigma}^{(N)}_{ij}(k,k') = \hat{P}^{(N)}_{ij}(k,k') - \hat{\Pi}^{(N)}_{ijk}\, \hat{\Pi}^{(N)}_{ijk'} \tag{4.7}$$

where

$$\hat{P}^{(N)}_{ij}(k,k') = \frac{1}{M^{(N)}_{ij}(k,k')} \sum_{n=1}^{N} Z^{(n)}_{ijk}\, Z^{(n)}_{ijk'}\, J^{(n)}_{ijk}\, J^{(n)}_{ijk'} \tag{4.8}$$

Indeed, since $M^{(N)}_{ij}(k,k') \to \infty$ as $N \to \infty$, $\hat{P}^{(N)}_{ij}(k,k') \xrightarrow{a.s.} P_{ij}(k,k')$ as $N \to \infty$ by the SLLN. Hence $\hat{\sigma}^{(N)}_{ij}(k,k') \xrightarrow{a.s.} \sigma_{ij}(k,k')$ as $N \to \infty$.

Thus, if the values of $M_{ij}^{(N)}(k,k')$ are large for all k and k', the estimates of $\sigma_{ij}(k,k')$ will be close to the true values, with high probability. If, however, some of the values of $M_{ijk}^{(N)}(k,k')$ are not sufficiently large some algebraic inconsistencies might occur in the values of the estimators since due to the missing values, $\hat{\Pi}_{ijk}^{(N)}$, $\hat{\Pi}_{ijk'}^{(N)}$ and $\hat{P}_{ij}^{(N)}(k,k')$ are not necessarily based on the same sample values. The following is an illustration of this point in a case of an extremely small sample. Suppose that we have three requirements with the following data.

	req.		
n	1	2	3
1	1	9	0
2	0	1	9
3	9	0	9
4	1	1	1
5	1	9	1
6	9	1	1

The statistics of this small sample are:

$$N = 6\ ,\ M_1^{(6)} = M_2^{(6)} = M_3^{(6)} = 4,\ \ \hat{\Pi}_1^{(6)} = \hat{\Pi}_2^{(6)} = \hat{\Pi}_3^{(6)} = .75\ ,$$

$$M_{12}^{(6)} = 2\ ,\ M_{13}^{(6)} = 3,\ M_{23}^{(6)} = 2\ ,\ \hat{P}_{12}^{(6)} = .50\ ,\ \hat{P}_{13}^{(6)} = .67\ ,$$

$\hat{P}_{23}^{(6)} = 1$. The corresponding estimator of (V) is

$$(\hat{V}) = \begin{bmatrix} .1875 & -.0625 & .1075 \\ -.0625 & .1875 & .4375 \\ .1075 & .4375 & .1875 \end{bmatrix}$$

This matrix is, however, not positive definite as $\hat{\sigma}^2(2,3) > \hat{\sigma}(2,2)\ \hat{\sigma}(3,3)$. This result is obviously inadmissible! Such algebraic inconsistencies, when covariances are estimated from incomplete data sets, have been previously mentioned in the literature (see Little (1979)).

In the following section we derive pair-wise maximum likelihood estimators (PMLE) of $P_{ij}(k,k')$ and of $P_{i\cdot jj'}(k,\ell)$. These PMLE yield algebraically consistent estimators of the covariances between any pair of requirements and also more efficient estimators of the marginal probabilities Π_{ijk} .

5. Pairwise Maximum Likelihood Estimators (PMLE)

Consider any two requirements, say (i.j.k) and (i,j'.ℓ) . We are interested in estimating $C_{i\cdot jj'}(k,\ell) = P_{i\cdot jj'}(k,\ell) - \Pi_{ijk}\ \Pi_{ij'\ell}$. If there were no missing values in the data set, i.e., if all $Z_{ijk}^{(n)} = 1$, then the covariance estimators presented in Section 4.2 are also maximum likelihood (MLE) and thus best asymptotically normal (BAN) (see Zacks [1981; pp. 249]) . This is not the case, however under missing value situations. It will be shown that since different requirements are not independent, the MLE of $P_{i\cdot jj'}(k,\ell)$ and of Π_{ijk} , $\Pi_{ij'\ell}$, utilize more of the information in the data than the unbiased estimators discussed in the previous section. The initial step in deriving the MLE's is the explicit (or implicit) formulation of the likelihood function of the parameters given the data. Due to the missing value problem, one should formulate these likelihood functions in terms of all the $\prod_{i=1}^{3}\prod_{j=1}^{t_i} 2^{r_{ij}}$ joint probabilities $\Pi(\ell_{111},\ldots,\ell_{1t_1r_{t_1}},\ldots,\ell_{311}\ldots,\ell_{3t_3r_{t_3}})$ for $\ell_{ijk} = 0\ ,\ 1$, of the possible realizations of the random variable $J_{111},\ldots,J_{3t_3r_{t_3}}$. Since we have only N=75 evaluations,

the data sets will have zero frequencies for most of the corresponding cells. We therefore resort to an ad-hoc solution, in which we estimate $P_{i\cdot jj'}(k,\ell)$ by the MLE based on the data available for each pair of requirements separately. These estimates are called <u>pairwise maximum likelihood</u> (PMLE). The PMLE of $P_{ijj'}(k,\ell)$ coincides with the MLE of $P_{ijj'}(k,\ell)$ when there are no missing values.

5.1 Derivation of the PMLE and their Asymptotic Covariance Matrix

5.1.1 The Likelihood Equation

Consider any two requirements $(i.j.k)$ and $(i.j'.\ell)$, within the i-th MPS. For the purpose of simplifying notation, we will disregard in the present section the actual indexes of these requirements, and label the corresponding variables by $Z_1^{(n)}$, $J_1^{(n)}$ and $Z_2^{(n)}$, $J_2^{(n)}$, $n=1,\ldots,N$. The sufficient statistic for the joint distributions of these variables are given in the following table.

Table 5.1

Frequency Distribution of $(Z_1,J_1)\&(Z_2,J_2)$

		$Z_2 = 1$		$Z_2 = 0$
Z_1	J_1	$J_2 = 0$	$J_2 = 1$	-
1	0	f_{00}	f_{01}	$g_{0\cdot}$
	1	f_{10}	f_{11}	$g_{1\cdot}$
0	-	$g_{\cdot 0}$	$g_{\cdot 1}$	h

where

$$f_{00} = \sum_{n=1}^{N} Z_1^{(n)} Z_2^{(n)} (1-J_1^{(n)})(1-J_2^{(n)}) \quad ,$$

$$f_{01} = \sum_{n=1}^{N} Z_1^{(n)} Z_2^{(n)} (1-J_1^{(n)}) J_2^{(n)} \quad ,$$

$$f_{10} = \sum_{n=1}^{N} Z_1^{(n)} Z_2^{(n)} J_1^{(n)}(1-J_2^{(n)}) \quad ,$$

$$f_{11} = \sum_{n=1}^{N} Z_1^{(n)} Z_2^{(n)} J_1^{(n)} J_2^{(n)} \quad ,$$

$$g_{0\cdot} = \sum_{n=1}^{N} Z_1^{(n)} (1-Z_2^{(n)}) (1-J_1^{(n)}) \quad ,$$

$$g_{1\cdot} = \sum_{n=1}^{N} Z_1^{(n)} (1-Z_2^{(n)}) J_1^{(n)}$$

$$g_{\cdot 0} = \sum_{n=1}^{N} (1-Z_1^{(n)}) Z_2^{(n)} (1-J_2^{(n)}) \quad ,$$

$$g_{\cdot 1} = \sum_{n=1}^{N} (1-Z_1^{(n)}) Z_2^{(n)} J_2^{(n)}$$

and

$$h = \sum_{n=1}^{N} (1-Z_1^{(n)})(1-Z_2^{(n)}) \quad .$$

Let $M = f_{00} + f_{01} + f_{10} + f_{11}$. Obviously, $N = M + g_{0\cdot} + g_{1\cdot} + g_{\cdot 0} + g_{\cdot 1} + h$. In the case of complete data $M = N$. The frequencies $g_{i\cdot}$ $(i=0,1)$ and $g_{\cdot 1}$ $(i=0,1)$ are of partially observed data. h is the number of evaluations in which both requirements are missing.

The cell probabilities are denoted by

$$\alpha = P\left\{ J_1^{(n)} = 0 \ , \ J_2^{(n)} = 0 \right\} \ ,$$

$$\beta = P\left\{ J_1^{(n)} = 0 \ , \ J_2^{(n)} = 1 \right\}$$

and

$$\gamma = P\left\{ J_1^{(n)} = 1 \ , \ J_2^{(n)} = 0 \right\} \ .$$

Accordingly, the kernel of the log-likelihood of (α,β,γ) is

$$\begin{aligned} \ell(\alpha,\beta,\gamma) = {} & f_{00} \log \alpha + f_{01} \log \beta + f_{10} \log \gamma \\ & + f_{11} \log (1-\alpha-\beta-\gamma) + g_{0\cdot} \log (\alpha + \beta) \\ & + g_{1\cdot} \log (1-\alpha-\beta) + g_{\cdot 0} \log (\alpha + \gamma) \\ & + g_{\cdot 1} \log (1-\alpha-\gamma) \ . \end{aligned} \tag{5.1}$$

Indeed, $\underset{\sim}{f} = (f_{00}, f_{01}, f_{10}, f_{11})$ has a multinomial distribution while $(g_{0\cdot} \ , \ g_{1\cdot})$ and $(g_{\cdot 0} \ , \ g_{\cdot 1})$ are binomial, independent of $\underset{\sim}{f}$.

5.1.2 Determination of PMLE by the EM-Algorithm

The vector $\hat{\underset{\sim}{\theta}} = (\hat{\alpha}, \hat{\beta}, \hat{\gamma})$, within the simplex $\Theta = \{\underset{\sim}{\theta} \ ; \ 0 \le \alpha,\beta,\gamma$ and $\alpha + \beta + \gamma \le 1\}$ which maximizes (5.1) is called the PMLE of $\underset{\sim}{\theta} = (\alpha, \beta, \gamma)$. Derivation of the PMLE can be obtained either by solving the set of nonlinear equations

$$(\nabla)_\theta \ell(\underset{\sim}{\theta}) = 0 \ , \tag{5.2}$$

or by applying the E-M algorithm of Dempster, Laird and Rubin (1977). The E-M algorithm requires several iterations, but the computations are simpler than those required to solve (5.2) iteratively. This is due to the fact that, in the case of N=M, the solution of (5.2) within Θ is simply

$$\hat{\alpha} = f^*_{00}/N\ , \quad \hat{\beta} = f^*_{01}/N\ , \quad \text{and} \quad \hat{\gamma} = f^*_{10}/N\ . \tag{5.3}$$

where f^*_{ij} (i,j=0,1) designates the cell frequencies based on the complete data. Generally, f^*_{ij} are unavailable. The E-M algorithm estimates these frequencies at each iteration, by estimating the missing data from the estimates of α, β, γ of the previous iteration. Thus if $\alpha^{(p)}$, $\beta^{(p)}$ and $\gamma^{(p)}$ denote the estimates of α, β and γ obtained after the p-th iteration, then the estimates of f^*_{ij} at the (p+1)st iteration are

$$\begin{aligned} f^{(p+1)}_{00} &= f_{00} + g_{0.}\ \alpha^{(p)}/(\alpha^{(p)} + \beta^{(p)}) \\ &\quad + g_{.0}\ \alpha^{(p)}/(\alpha^{(p)} + \gamma^{(p)}) + h\ \alpha^{(p)}\ , \\ f^{(p+1)}_{01} &= f_{01} + g_{0.}\ \beta^{(p)}/(\alpha^{(p)} + \beta^{(p)}) \\ &\quad + g_{.1}\ \beta^{(p)}/(1-\alpha^{(p)} - \gamma^{(p)}) + h\ \beta^{(p)}\ , \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} f^{(p+1)}_{10} &= f_{10} + g_{1.}\ \gamma^{(p)}/(1-\alpha^{(p)} - \beta^{(p)}) \\ &\quad + g_{.0}\ \gamma^{(p)}/(\alpha^{(p)} + \gamma^{(p)}) + h\ \gamma^{(p)}\ . \end{aligned}$$

The estimates of α, β and γ are changed then to $\alpha^{(p+1)}$, $\beta^{(p+1)}$ and $\gamma^{(p+1)}$, by substituting $f_{ij}^{(p+1)}$ for f_{ij}^{*} $(i,j=0,1)$ in (5.3). The initial estimates in this process are

$$f_{ij}^{(0)} = (f_{ij} + .5)/(M + 2) \qquad , \ i,j=0,1 . \tag{5.5}$$

These initial estimates will assure that at each iteration $\alpha^{(p)}$, $\beta^{(p)}$ and $\gamma^{(p)}$ are positive and $f_{ij}^{(p+1)}$ are always determinable. Notice that formulae (5.4) provide estimates of the conditional expectation of f_{ij}^{*} given the sufficient statistic $(\underset{\sim}{f}\ , g_{0\cdot}, g_{1\cdot}, g_{\cdot 0}, g_{\cdot 1})$. This is the E (expectation) phase of the algorithm.

Substitution of $f_{ij}^{(p+1)}$ in (5.3) provides the M-phase estimates of $\underset{\sim}{\theta}$. The following numerical example illustrates this algorithm and its convergence.

Table 5.2

Frequency Distribution

		$Z_2 = 1$		$Z_2 = 0$
Z_1	J_1	$J_2 = 0$	$J_2 = 1$	-
1	0	20	10	5
	1	5	15	5
0	-	5	5	5

Table 5.3

Iterative PMLE

p	$\alpha^{(p)}$	$\beta^{(p)}$	$\gamma^{(p)}$
0	.3942	.2019	.1058
1	.3896	.1963	.1053
2	.3894	.1947	.1048
3	.3896	.1940	.1045
4	.3897	.1940	.1045
5	.3897	.1940	.1045

Convergence to within $\varepsilon \leq 5 \times 10^{-5}$ was attained in five iterations.

5.1.3 The Asymptotic Variance-Covariance Matrix of the PMLE

Let $M_{1\cdot} = g_{0\cdot} + g_{1\cdot}$ and $M_{\cdot 1} = g_{\cdot 0} + g_{\cdot 1}$. These are the number of observations on one variable in which the values of the other variables are missing. The score functions corresponding to (5.1) are defined as

$$(5.6) \qquad S_1 = \frac{\partial \ell(\alpha,\beta,\gamma)}{\partial \alpha}$$

$$= \frac{f_{00}}{\alpha} - \frac{f_{11}}{1-\alpha-\beta-\gamma} + \frac{g_{0\cdot}}{(\alpha+\beta)(1-\alpha-\beta)} - \frac{M_{1\cdot}}{1-\alpha-\beta}$$

$$+ \frac{g_{\cdot 0}}{(\alpha+\gamma)(1-\alpha-\gamma)} - \frac{M_{\cdot 1}}{1-\alpha-\gamma} ,$$

$$(5.7) \qquad S_2 = \frac{\partial \ell(\alpha,\beta,\gamma)}{\partial \beta}$$

$$= \frac{f_{01}}{\beta} - \frac{f_{11}}{1-\alpha-\beta-\gamma} + \frac{g_{0\cdot}}{(\alpha+\beta)(1-\alpha-\beta)} - \frac{M_{1\cdot}}{1-\alpha-\beta}$$

and

$$(5.8) \qquad S_3 = \frac{\partial \ell(\alpha,\beta,\gamma)}{\partial \gamma}$$

$$= \frac{f_{10}}{\gamma} - \frac{f_{11}}{1-\alpha-\beta-\gamma} + \frac{g_{\cdot 0}}{(\alpha+\gamma)(1-\alpha-\gamma)} - \frac{M_{\cdot 1}}{1-\alpha-\gamma}$$

Finally, since the family of density functions of the minimal sufficient statistic is smooth in every open neighborhood within Θ the Fisher information matrix (FIM) of (α,β,γ) is given by the variance-covariance matrix of $\underset{\sim}{S} = (S_1, S_2, S_3)$. Straighforward manipulations yield the FIM,

$$(5.9) \qquad (II(\underset{\sim}{\theta})) = M \begin{bmatrix} A\frac{1-\beta-\gamma}{\alpha} + B+C & A+B & A+C \\ A+B & A\frac{1-\alpha-\gamma}{\beta} + B & A \\ A+C & A & A\frac{1-\alpha-\beta}{\gamma} + C \end{bmatrix}$$

where

$$A = \frac{1}{1-\alpha-\beta-\gamma} \quad ,$$

$$(5.10) \qquad B = \frac{M_{1\cdot}}{M(\alpha+\beta)(1-\alpha-\beta)} \quad ,$$

$$C = \frac{M_{\cdot 1}}{M(\alpha+\gamma)(1-\alpha-\gamma)} \quad .$$

Thus, the asymptotic distribution of the PLME, $\hat{\underset{\sim}{\theta}}$, is normal with mean $\underset{\sim}{\theta}$ and variance-covariance matrix $\frac{1}{M}(V(\underset{\sim}{\theta}))$, where $(V(\underset{\sim}{\theta})) = (II(\underset{\sim}{\theta}))^{-1}$.

Explicit expressions for the elements of $(V(\underset{\sim}{\theta}))$ are:

$$V_{11}(\underset{\sim}{\theta}) = \frac{1}{\mathrm{Det}} \left[(A \frac{1-\alpha-\gamma}{\beta} + B) \cdot (A \frac{1-\alpha-\beta}{\gamma} + C) - A^2 \right],$$

$$V_{12}(\underset{\sim}{\theta}) = \frac{-1}{\mathrm{Det}} \left[(A+B) \cdot (A \frac{1-\alpha-\beta}{\gamma} + C) - A(A+C) \right],$$

$$(5.11) \quad V_{13}(\underset{\sim}{\theta}) = \frac{1}{\mathrm{Det}} \left[A(A+B) - (A+C) \cdot (A \frac{1-\alpha-\gamma}{\beta} + B) \right],$$

$$V_{22}(\underset{\sim}{\theta}) = \frac{1}{\mathrm{Det}} \left[(A \frac{1-\beta-\gamma}{\alpha} + B+C) (A \frac{1-\alpha-\beta}{\gamma} + C) - (A+C)^2 \right],$$

$$V_{23}(\underset{\sim}{\theta}) = \frac{-1}{\mathrm{Det}} \left[(A \frac{1-\beta-\gamma}{\alpha} + B+C) A - (A+B) \cdot (A+C) \right],$$

$$V_{33}(\underset{\sim}{\theta}) = \frac{1}{\mathrm{Det}} \left[(A \frac{1-\beta-\gamma}{\alpha} + B+C) (A \frac{1-\alpha-\gamma}{\beta} + B) - (A+B)^2 \right],$$

and

$$(5.12) \quad \mathrm{Det} = \left[A \frac{1-\beta-\gamma}{\alpha} + B+C \right] \cdot \left[(A \frac{1-\alpha-\gamma}{\beta} + B) \cdot (A \frac{1-\alpha-\beta}{\gamma} + C) - A^2 \right] - (A+B) \cdot \left[(A+B) \cdot (A \frac{1-\alpha-\beta}{\gamma} + C) - A(A+C) \right] + (A+C) \cdot \left[A(A+B) - (A \frac{1-\alpha-\gamma}{\beta} + B) (A+C) \right].$$

PMLE of the matrix $(V(\underset{\sim}{\theta}))$ is obtained by substituting the PMLE's of α, β, γ in formulae (5.11) and (5.12).

In Table 5.5 we present numerical values of the PMLE corresponding to the frequency distribution of Table 5.4.

Table 5.4

Frequency Distribution of (Z_1, J_1) and (Z_2, J_2)

Z_1	J_1	$Z_2 = 1$, $J_2 = 0$	$Z_2 = 1$, $J_2 = 1$	$Z_2 = 0$
1	0	5	5	0
1	1	10	40	5
0	-	5	5	0

Table 5.5

PMLE of (α,β,γ) and of $V(\underset{\sim}{\theta})$ corresponding to Data of Table 5.4

	α	β	γ
$\underset{\sim}{\theta}$	.0977	.1122	.1456
$V(\underset{\sim}{\theta})$	.0013556	-.0002083	-.0003225
	.	.0015880	-.0002094
	.	.	.0019071

5.2 Combined PMLE of ECTS

As in the previous section, consider any two requirements in a given task, and let $(\underset{\sim}{f}\ ,\ g_{0\cdot}\ ,\ M_{1\cdot}\ ,\ g_{\cdot 0}\ ,\ M_{\cdot 1})$ be the frequency distribution of the response (the sufficient statistic) and let $\underset{\sim}{\theta} = (\alpha,\beta,\gamma)$ be the cell probabilities. Let $\Pi_{1\cdot} = 1-\alpha-\beta$ be the probability of $\{J_1^{(n)} = 1\}$ and $\Pi_{\cdot 2} = 1-\alpha-\gamma$ the probability of $\{J_2^{(n)} = 1\}$. $\Pi_{1\cdot}$ and $\Pi_{\cdot 1}$ represent Π_{ijk} and $\Pi_{ijk'}$, respectively.

The unbiased estimator of Π_{ijk} , given by (4.1), can be represented in the present framework by the formula

$$\tilde{\Pi}_{1\cdot} = \frac{f_{10} + f_{11} + M_{1\cdot} - g_{0\cdot}}{M + M_{1\cdot}} \tag{5.13}$$

The variance of this estimator is

$$V\ \{\tilde{\Pi}_{1\cdot}\} = (\alpha+\beta)(1-\alpha-\beta)/(M+M_{1\cdot}) \tag{5.14}$$

Similarly,

$$V\ \{\tilde{\Pi}_{\cdot 1}\} = (\alpha+\gamma)(1-\alpha-\gamma)/(M+M_{\cdot 1})\ . \tag{5.15}$$

The PMLE are, on the other hand,

$$\begin{aligned} \hat{\Pi}_{1\cdot} &= 1-\hat{\alpha}-\hat{\beta} \\ \hat{\Pi}_{\cdot 1} &= 1-\hat{\alpha}-\hat{\gamma} \end{aligned} \tag{5.16}$$

having asymptotic variances of

$$\begin{aligned} AV\ \{\hat{\Pi}_{1\cdot}\} &= \frac{1}{M}\ [V_{11}(\underset{\sim}{\theta}) + V_{22}(\underset{\sim}{\theta}) + 2\ V_{12}(\underset{\sim}{\theta})\ ] \\ AV\ \{\hat{\Pi}_{\cdot 1}\} &= \frac{1}{M}\ [V_{11}(\underset{\sim}{\theta}) + V_{33}(\underset{\sim}{\theta}) + 2\ V_{13}(\underset{\sim}{\theta})]\ , \end{aligned} \tag{5.17}$$

where $V_{ij}(\underset{\sim}{\theta})$ are specified in (5.11). To illustrate the difference between the unbiased and the PMLE, we compare the asymptotic variances of $\tilde{\Pi}_{1\cdot}$ $(\tilde{\Pi}_{\cdot 1})$ and of $\hat{\Pi}_{1\cdot}$ $(\hat{\Pi}_{\cdot 1})$, with the estimates of the parameters given in Table 5.5.

Table 5.6

Asymptotic Variances of Estimates of $\Pi_{1\cdot}$ $(\Pi_{\cdot 1})$ and their relative efficiency

Parameter	PMLE	Unbiased	ARE
$\Pi_{1\cdot}$	.002527	.002551	.990
$\Pi_{\cdot 1}$	.002618	.002832	.924

The asymptotic relative efficiency (ARE) of the unbiased estimator, $\tilde{\Pi}_{1\cdot}$, compared to the PMLE, $\hat{\Pi}_{1\cdot}$, is defined as the Pitman efficiency

$$\text{ARE} = \text{AV}\,\{\hat{\Pi}_{1\cdot}\}\,/\text{AV}\,\{\tilde{\Pi}_{1\cdot}\}\,. \tag{5.18}$$

In Table 5.6 we see that the ARE of $\tilde{\Pi}_{1\cdot}$ compared to $\hat{\Pi}_{1\cdot}$ is 99% and that of $\tilde{\Pi}_{\cdot 1}$ is 92%. This table indicates that due to the dependence between different requirements the PMLE might be slightly more efficient than the unbiased estimator.

Returning to the structure of the data set, consider task (i.j). There are r_{ij} requirements. Let $\hat{\Pi}_{ijk(\ell)}$ be the PMLE of Π_{ijk} obtained from the joint frequency distribution of the k-th and the ℓ-th requirement in this task $(k \neq \ell)$. Let $L_{ij} = r_{ij} - 1$. Since there are L_{ij} different such joint distributions, we consider a weighted

average of the L_{ij} different PMLE's of Π_{ijk} as an estimator. Accordingly, we define the estimator

$$(5.19) \qquad \hat{\hat{\Pi}}_{ijk} = \sum_{\ell=1}^{L_{ij}} M_{ij}(k,\ell)\, \hat{\Pi}_{ijk(\ell)} \Big/ \sum_{\ell=1}^{L_{ij}} M_{ij}(k,\ell) .$$

The variance of $\hat{\hat{\Pi}}_{ijk}$ is

$$(5.20) \qquad V\{\hat{\hat{\Pi}}_{ijk}\} = \frac{1}{\left(\sum_{\ell=1}^{L_{ij}} M_{ij}(k,\ell)\right)^2} \left\{ \sum_{\ell=1}^{L_{ij}} M_{ij}^2(k,\ell)\, V\{\hat{\Pi}_{ijk(\ell)}\} + 2\sum_{\ell < \ell'}\sum M_{ij}(k,\ell)\, M_{ij}(k,\ell')\, \mathrm{cov}\,(\hat{\Pi}_{ijk(\ell)}, \hat{\Pi}_{ijk(\ell')}) \right\}$$

In order to compute the asymptotic covariance between $\hat{\Pi}_{ijk(\ell)}$ and $\hat{\Pi}_{ijk(\ell')}$ one needs to extend the theoretical framework of the PMLE, to that of a simultaneous estimation of $\hat{\Pi}_{ijk(\ell)}$ and $\hat{\Pi}_{ijk(\ell')}$. An upper bound for (5.20) can be obtained by employing the inequality

$$(5.21) \qquad \left| \mathrm{cov}\,(\hat{\Pi}_{ijk(\ell)}, \hat{\Pi}_{ijk(\ell')}) \right| \le (V\{\hat{\Pi}_{ijk(\ell)}\}\, V\{\hat{\Pi}_{ijk(\ell')}\})^{1/2}$$

This inequality yields the following upper bound on the asymptotic variance of $\hat{\hat{\Pi}}_{ijk}$, namely:

$$(5.22) \qquad AV\{\hat{\hat{\Pi}}_{ijk}\} \le \frac{1}{\left(\sum_{\ell=1}^{L_{ij}} M_{ij}(k,\ell)\right)^2} \left\{ \sum_{\ell=1}^{L_{ij}} M_{ij}^2(k,\ell)\, AV\{\hat{\Pi}_{ijk(\ell)}\} + 2\sum_{\ell < \ell'}\sum M_{ij}(k,\ell)\, M_{ij}(k,\ell')\, (AV\{\hat{\Pi}_{ijk(\ell)}\}\, AV\{\hat{\Pi}_{ijk(\ell')}\})^{1/2} \right\}$$

$$= \left(\frac{\sum_{\ell=1}^{L_{ij}} M_{ij}(k,\ell) \, (AV\{\hat{\Pi}_{ijk(\ell)}\})^{1/2}}{\sum_{\ell=1}^{L_{ij}} M_{ij}(k,\ell)} \right)^2 .$$

$\hat{\hat{\Pi}}_{ijk}$ will be called the combined pairwise MLE (CPMLE) of Π_{ijk} .

The CPMLE of the ECTS, $\tau_{ij} = \sum_{k=1}^{r_{ij}} w_{ijk} \Pi_{ijk}$, is

$$\hat{\hat{\tau}}_{ij} = \sum_{k=1}^{r_{ij}} w_{ijk} \hat{\hat{\Pi}}_{ijk} . \tag{5.23}$$

An upper bound for the variance of $\hat{\hat{\tau}}_{ij}$ is

$$\begin{aligned} \mathrm{Var}\ \{\hat{\hat{\tau}}_{ij}\} &\leq \sum_{k=1}^{r_{ij}} w_{ijk}^2 \ V^* \{\hat{\hat{\Pi}}_{ijk}\} + \\ &\quad 2 \sum_{k < k'} \sum w_{ijk} w_{ijk'} \left[V^* \{\hat{\hat{\Pi}}_{ijk}\} \ V^* \{\hat{\hat{\Pi}}_{ijk'}\} \right]^{1/2} \\ &= \left(\sum_{k=1}^{r_{ij}} w_{ijk} \ SE^* \{\hat{\hat{\Pi}}_{ijk}\} \right)^2 \end{aligned} \tag{5.24}$$

where $V^* \{\hat{\hat{\Pi}}_{ijk}\}$ is given by the RHS of (5.22), and $SE^*\{\hat{\hat{\Pi}}_{ijk}\}$ is the square root of $V^* \{\hat{\hat{\Pi}}_{ijk}\}$ (an upper bound for the standard-error) . The RHS of (5.24) is denoted by V_{ij}^* . In Section 6 we provide numerical results, based on the actual data, which illustrate the effectiveness of using $\hat{\hat{\Pi}}_{ijk}$.

5.3 CPMLE of Variances and PMLE of Covariances

In the present section we discuss the estimation of the variances and covariances of $\underset{\sim}{X}_{ij}$, namely σ_{ij}^2 and $C_i(j,j')$. According to (3.5) a CPMLE estimator of σ_{ij}^2 is

(5.26) $$\hat{\hat{\sigma}}^2_{ij} = \underset{\sim}{w}'_{ij} \, (\hat{\hat{V}}_{ij}) \, \underset{\sim}{w}_{ij} \quad ,$$

where the elements of $(\hat{\hat{V}}_{ij})$ are

(5.27) $$\hat{\hat{\sigma}}_{ij}(k,k') = \begin{cases} \hat{\hat{\Pi}}_{ijk}(1-\hat{\hat{\Pi}}_{ijk}) & , \; k=k' \\ \hat{P}_{ij}(k,k') - \hat{\Pi}_{ijk(k')} \, \hat{\Pi}_{ijk'(k)} & , \; k \neq k' \end{cases}$$

where $\hat{P}_{ij}(k,k')$ is the PMLE of $P_{ij}(k,k')$ and $\hat{\Pi}_{ijk(k')}$, $\hat{\Pi}_{ijk'(k)}$ are the corresponding PMLE of Π_{ijk} and $\Pi_{ijk'}$, respectively. Thus, the PMLE of the $C_i(j,j')$ is

(5.28) $$\hat{C}_i(j,j') = \underset{\sim}{w}'_{ij} \, (\hat{K}_{i \cdot jj'}) \, \underset{\sim}{w}_{ij'} \quad ,$$

where the elements of the covariance matrix $\hat{K}_{i \cdot jj'}$ are

(5.29) $$\hat{C}_{i \cdot jj'}(k,\ell) = \hat{P}_{i \cdot jj'}(k,\ell) - \hat{\Pi}_{ijk(\ell)} \, \hat{\Pi}_{ij'\ell(k)} \; .$$

Finally, the correlations between complete task scores are estimated by the formula

(5.30) $$\hat{\rho}_{i \cdot jj'} = \hat{C}_i(j,j') / \hat{\hat{\sigma}}_{ij} \, \hat{\hat{\sigma}}_{ij'} \; .$$

5.4 PMLE Adjustment of Task Scores

The complete task score $\tilde{X}^{(n)}_{ij}$ of the n-th evaluation is unavailable if some of the requirements of that task have not been observed. One can, however, estimate the scores of the non-available requirements on the basis of the scores of the available requirements

and the estimates of the $P_{ij}(k,\ell)$ in that population. The CTS of task i.j at the n-th evaluation is

$$(5.31)\qquad \tilde{X}_{ij}^{(n)} = \sum_{k=1}^{r_{ij}} W_{ijk}\, Z_{ijk}^{(n)}\, J_{ijk}^{(n)} + \sum_{k=1}^{r_{ij}} W_{ijk}\, (1-Z_{ijk}^{(n)})\, J_{ijk}^{(n)}$$

An optimal predictor of $\tilde{X}_{ij}^{(n)}$, for the square-error loss function, is the conditional expectation of $\tilde{X}_{ij}^{(n)}$ given the data, i.e.,

$$(5.32)\qquad Y_{ij}^{(n)} = \sum_{k=1}^{r_{ij}} W_{ijk}\, Z_{ijk}^{(n)}\, J_{ijk}^{(n)} + \sum_{k=1}^{r_{ij}} W_{ijk}\, (1-Z_{ijk}^{(n)})\, E\{J_{ijk}^{(n)} | \mathcal{S}\},$$

where $E\{J_{ijk}^{(n)} | \mathcal{S}\}$ is the conditional expectation of $J_{ijk}^{(n)}$ given the sample data $\mathcal{S}$. This conditional expectation cannot be determined, since it requires the knowledge of all the possible joint probabilities $\pi(\ell_{111},\ldots,\ell_{3t_3\gamma_{t_3}})$. One can determine, however, estimates of the conditional expectations of $J_{ijk}^{(n)}$, given particular observed requirements by applying the PMLE of $P_{ij}(k,\ell)$ and $P_{ijj'}(k,\ell')$. Weighted averages of such estimates can be considered, with weights which are inversely proportional to the asymptotic variances of these estimates.

Let $A_{ij}^{(n)}$ denote the set of indexes of all available requirements of task i.j at the n-th evaluation. Let $E_{ijk}(J_{ij\ell}^{(n)})$ denote the conditional expectation of $J_{ijk}^{(n)}$, given $J_{ij\ell}^{(n)}$, where $\ell \in A_{ij}^{(n)}$. The PMLE of $E_{ijk}(J_{ij\ell}^{(n)})$ is

$$(5.33)\qquad \hat{E}_{ijk}(J_{ij\ell}^{(n)}) = J_{ij\ell}^{(n)}\, \hat{P}_{ij}(k,\ell)/\hat{\pi}_{ij\ell(k)} + (1-J_{ij\ell}^{(n)})\, [\hat{\pi}_{ijk(\ell)} - \hat{P}_{ij}(k,\ell)]/(1-\hat{\pi}_{ij\ell(k)}),$$

in which $\hat{\pi}_{ijk(\ell)}$ and $\hat{\pi}_{ij\ell(k)}$ are the PMLE of π_{ijk} and of $\pi_{ij\ell}$, respectively, obtained from the sample data of requirements i.j.k and i.j.ℓ.

As in Section 5.1.1, let $\alpha = P[J^{(n)}_{ijk} = 0 , J^{(n)}_{ij\ell} = 0]$, $\beta = P[J^{(n)}_{ijk} = 0 , J^{(n)}_{ij\ell} = 1]$ and $\gamma = P[J^{(n)}_{ijk} = 1 , J^{(n)}_{ij\ell} = 0]$. Accordingly, $P_{ij}(k,\ell) = 1-\alpha-\beta-\gamma$ and $\pi_{ijk(\ell)} = 1-\alpha-\beta$, $\pi_{ij\ell(k)} = 1-\alpha-\gamma$. Thus,

$$\hat{E}_{ijk}\,(J^{(n)}_{ij\ell}) = J^{(n)}_{ij\ell}\,(1-\frac{\hat{\beta}}{1-\hat{\alpha}-\hat{\gamma}}) + (1-J^{(n)}_{ij\ell})\,\frac{\hat{\gamma}}{\hat{\alpha}+\hat{\gamma}} \tag{5.34}$$

Furthermore, let $F^{(n)}_{ijk(\ell)}\,(\alpha,\beta,\gamma) = J^{(n)}_{ij\ell}\,(1-\frac{\beta}{1-\alpha-\gamma}) + (1-J^{(n)}_{ij\ell})\,\frac{\gamma}{\alpha+\gamma}$.

The asymptotic conditional variance of $\hat{E}_{ijk}\,(J^{(n)}_{ij\ell})$, given $J^{(n)}_{ij\ell}$, is, approximately, given by

$$AV\{\hat{E}_{ijk}(J^{(n)}_{ij\ell})\,|\,J^{(n)}_{ij\ell}\} \cong \frac{1}{M}\,(\underset{\sim}{\nabla}^{(n)}_{ijk(\ell)})'\,(V(\underset{\sim}{\Theta}))\,(\underset{\sim}{\nabla}^{(n)}_{ijk(\ell)}) , \tag{5.35}$$

where

$$\begin{aligned}(\underset{\sim}{\nabla}^{(n)}_{ijk(\ell)})' &= (\frac{\partial}{\partial\alpha}F^{(n)}_{ijk(\ell)}, \frac{\partial}{\partial\beta}F^{(n)}_{ijk(\ell)}, \frac{\partial}{\partial\gamma}F^{(n)}_{ijk(\ell)}) \\ &= -J^{(n)}_{ij\ell}\left(\frac{\beta}{(1-\alpha-\gamma)^2}-\frac{\gamma}{(\alpha+\gamma)^2}, \frac{1}{(1-\alpha-\gamma)}, \frac{\beta}{(1-\alpha-\gamma)^2}+\frac{\alpha}{(\alpha+\gamma)^2}\right) - \left(\frac{\gamma}{(\alpha+\gamma)^2}, 0, \frac{-\alpha}{(\alpha+\gamma)^2}\right).\end{aligned} \tag{5.36}$$

Since the N different evaluations are independent, the dependence between the MLE's $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ and a particular value $J_{ij\ell}^{(n)}$ becomes negligible when N grows. Formulae (5.35) - (5.36) are valid for the case of $J_{ij\ell}^{(n)}$ independent of $(\hat{\alpha},\hat{\beta},\hat{\gamma})$.

Finally, let $\hat{D}_{ijk(\ell)}^{(n)}$ denote the asymptotic variance of the PMLE of (5.35). A predictor of $J_{ijk}^{(n)}$, based on the values of the available requirements in task (i.j) is the weighted average

$$(5.37) \qquad \hat{E}_{ijk} = \frac{1}{T_{ijk}^{(n)}} \sum_{\ell \in A_{ij}^{(n)}} \hat{E}_{ijk}(J_{ij\ell}^{(n)})/\hat{D}_{ijk(\ell)}^{(n)} ,$$

where

$$T_{ijk}^{(n)} = \sum_{\ell \in A_{ij}^{(n)}} 1/\hat{D}_{ijk(\ell)}^{(n)} .$$

The weights in (5.37) are inversely proportional to the conditional variances of the PMLE. A numerical example of such prediction of missing observations is given in Section 6.

6. Numerical Results

In the present section we provide a few tables which demonstrate the theory developed in the previous sections. The results presented here are based on available data from N=75 evaluations of infantry battalions. For obvious reasons we do not present here the results explicitly or completely, but only fragmentally. In Table 6.1 we provide estimates of the probabilities of satisfying requirements, corresponding to four tasks labeled a, b, c, and d. The five rows corresponding to each case present, respectively, the unbiased estimators, the CPMLE, the number of applicable evaluations for the requirement $(M_{ijk}^{(n)})$, the estimated standard error (SE) of the unbiased estimator $[\hat{\sigma}_{ij}(k,k)/M_{ij}(k,k)]^{1/2}$, and the upper bound for the SE of the CPMLE, as obtained from formula (5.22).

As seen in the table, even in a case with a considerable number of missing values, the unbiased estimators and the CPMLE mostly yield identical results. The SE's are also very close. As shown earlier, the CPMLE might be somewhat more precise than the unbiased estimators, but the difference is not consequential. The values in Table 6.1 are estimated upper bounds for the SE's of the CPMLE.

In Table 6.2 we present the estimates of the ECTS and estimates of their SE's. Both the unbiased and the CPMLE are presented. Again, the values obtained by both estimators are very close. The values in the SE column of the CPMLE are the upper bounds as in formulae (5.24). These upper bounds are of no value. The precision of the CPMLE should be somewhat better than that of the unbiased estimator.

In Table 6.3 we provide estimates of the intertask correlations. These are the MLE's. The table consists only of the twelve tasks in MPS 1. It is interesting to notice that all the negative correlations are small in magnitude. Based on an ordinary t-test, the negative correlations are not significantly different from zero. On the other hand, some tasks show considerable positive correlations. This result is intuitively expected.

Finally, in Table 6.4 we present predictors of missing requirement scores in one task, as computed by formula (5.37). Each row in this table corresponds to a different evaluation. The values 1.00 or 0.0 are the actual scores of applicable requirements. Values between 0 and 1 are estimates of the conditional probability of satisfying a non-applicable (missing) requirement, given the scores of the applicable requirements in that evaluation. A row consisting of 9's indicates that the whole task was not applicable. The above examples illustrate that the methods developed in the present paper can be easily implemented on a large size computer.

Table 6.1

Unbiased Estimators and CPMLE of
π_{ijk} , and Their Estimated Standard Errors

Tasks	Requirements								
	1	2	3	4	5	6	7	8	9
a	0.878	0.587	0.919	0.730	0.973	0.880	0.733	0.907	0.747
	0.878	0.587	0.919	0.730	0.973	0.880	0.733	0.907	0.747
	74	75	74	74	75	75	75	75	75
	0.038	0.057	0.032	0.052	0.019	0.038	0.051	0.034	0.050
	0.038	0.057	0.032	0.052	0.019	0.038	0.051	0.034	0.050
b	0.507	0.773	0.733	0.932	0.947				
	0.507	0.773	0.733	0.929	0.947				
	75	75	75	74	75				
	0.058	0.048	0.051	0.029	0.026				
	0.058	0.048	0.051	0.030	0.026				
c	0.887	0.959	0.840	0.980	0.960	0.959	0.833	0.857	0.979
	0.887	0.959	0.840	0.980	0.960	0.960	0.833	0.839	0.979
	53	49	50	50	50	49	48	21	48
	0.044	0.028	0.052	0.020	0.028	0.028	0.054	0.076	0.021
	0.044	0.028	0.052	0.020	0.028	0.028	0.054	0.078	0.021
d	0.865	0.960	0.959	0.753	0.919	0.987	0.878	0.987	
	0.865	0.960	0.959	0.754	0.919	0.987	0.878	0.987	
	74	75	74	73	74	75	74	75	
	0.040	0.023	0.023	0.050	0.032	0.013	0.038	0.013	
	0.040	0.023	0.023	0.050	0.032	0.013	0.038	0.013	

Table 6.2

Unbiased and CPMLE Estimates of
The ECTS and their Estimated SE

Task	Estimators		SE	
	Unbiased	CPMLE	Unbiased	CPMLE
a	0.8146	0.8146	0.0198	0.0417
b	0.7489	0.7485	0.0281	0.0449
c	0.9255	0.9253	0.0180	0.0343
d	0.9187	0.9188	0.0150	0.0280

Table 6.3

MLE of the Inter-task Correlations
in MPS 1

	1	2	3	4	5	6	7	8	9	10	11	12
1	1.00	0.47	0.50	0.52	0.32	0.13	0.05	0.29	0.41	0.32	-0.06	0.37
2	0.47	1.00	0.39	0.33	0.17	0.21	0.25	0.02	0.11	0.04	0.08	0.22
3	0.50	0.39	1.00	0.51	0.42	0.27	0.31	0.06	0.19	0.19	-0.19	0.19
4	0.52	0.33	0.51	1.00	0.40	0.44	0.38	0.06	0.37	0.53	0.05	0.27
5	0.32	0.17	0.42	0.40	1.00	0.67	0.66	0.30	0.38	0.18	0.21	0.35
6	0.13	0.21	0.27	0.44	0.67	1.00	0.65	0.06	0.19	0.13	0.19	0.42
7	0.05	0.25	0.31	0.38	0.66	0.65	1.00	-0.08	0.28	0.23	0.25	0.13
8	0.29	0.02	0.06	0.06	0.30	0.06	-0.08	1.00	0.21	-0.02	0.07	0.07
9	0.41	0.11	0.19	0.37	0.38	0.19	0.28	0.21	1.00	0.21	-0.02	0.08
10	0.32	0.04	0.19	0.53	0.18	0.13	0.23	-0.02	0.21	1.00	-0.15	0.29
11	-0.06	0.08	-0.19	0.05	0.21	0.19	0.25	0.07	-0.02	-0.15	1.00	0.21
12	0.37	0.22	0.19	0.27	0.35	0.42	0.13	0.07	0.08	0.29	0.21	1.00

Table 6.4

Observed and Predicted Requirement
Scores in One Task

1.00	1.00	1.00	0.0	0.0	1.00	0.0	0.0	1.00	1.00
0.0	0.0	1.00	1.00	0.0	0.0	1.00	1.00	1.00	1.00
1.00	1.00	1.00	1.00	0.0	0.75	0.0	0.69	1.00	1.00
1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.0	1.00	1.00
1.00	1.00	1.00	1.00	1.00	0.76	1.00	0.70	0.0	0.0
1.00	1.00	1.00	1.00	1.00	0.75	0.0	0.69	0.0	0.0
1.00	1.00	1.00	0.93	0.69	0.76	0.65	0.70	0.88	0.87
9.00	9.00	9.00	9.00	9.00	9.00	9.00	9.00	9.00	9.00
1.00	1.00	1.00	0.93	0.69	0.76	0.65	0.70	0.88	0.87
9.00	9.00	9.00	9.00	9.00	9.00	9.00	9.00	9.00	9.00
1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.0	1.00
9.00	9.00	9.00	9.00	9.00	9.00	9.00	9.00	9.00	9.00
1.00	1.00	0.0	1.00	1.00	0.0	1.00	0.0	1.00	1.00
1.00	1.00	1.00	1.00	0.0	0.0	1.00	0.0	1.00	1.00
1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.00	1.00	1.00	1.00	1.00	1.00	0.0	0.0	1.00	1.00
1.00	1.00	1.00	1.00	0.0	0.0	1.00	0.0	1.00	1.00
1.00	1.00	1.00	1.00	0.0	0.0	0.0	0.0	1.00	0.0
1.00	1.00	1.00	1.00	0.0	0.0	0.0	1.00	1.00	1.00
1.00	1.00	1.00	1.00	0.0	0.0	1.00	1.00	1.00	1.00
1.00	1.00	1.00	1.00	1.00	0.0	0.0	0.0	1.00	1.00
1.00	1.00	1.00	1.00	1.00	0.76	1.00	1.00	1.00	1.00
9.00	9.00	9.00	9.00	9.00	9.00	9.00	9.00	9.00	9.00
1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
9.00	9.00	9.00	9.00	9.00	9.00	9.00	9.00	9.00	9.00
1.00	1.00	1.00	1.00	1.00	0.0	0.0	0.0	0.0	1.00
1.00	1.00	1.00	1.00	1.00	1.00	0.0	0.0	1.00	1.00
1.00	1.00	1.00	0.0	0.69	0.76	0.64	0.69	0.98	0.0
1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.00	1.00	1.00	1.00	0.0	1.00	0.0	0.0	1.00	0.0
1.00	1.00	1.00	1.00	0.0	0.0	1.00	0.69	1.00	1.00
0.0	1.00	1.00	1.00	0.0	0.0	0.0	1.00	1.00	1.00
1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.71	1.00	1.00
1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.0	1.00	0.0	1.00	0.0	1.00	0.0	0.0	1.00	1.00
1.00	1.00	1.00	1.00	1.00	0.76	0.0	1.00	1.00	1.00
1.00	1.00	1.00	1.00	0.0	1.00	0.0	0.0	1.00	1.00
1.00	1.00	1.00	1.00	0.0	1.00	1.00	1.00	1.00	1.00
0.0	1.00	1.00	1.00	0.0	1.00	0.0	1.00	1.00	1.00
1.00	1.00	0.0	1.00	0.0	1.00	0.0	1.00	1.00	1.00
1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.00	1.00	1.00	1.00	0.0	1.00	1.00	0.70	1.00	1.00
1.00	0.0	0.0	1.00	1.00	0.0	0.0	0.0	1.00	1.00
1.00	1.00	1.00	1.00	0.0	1.00	1.00	1.00	1.00	1.00
1.00	1.00	1.00	1.00	1.00	0.75	0.65	0.70	1.00	1.00
1.00	1.00	1.00	1.00	1.00	1.00	0.0	1.00	1.00	1.00
1.00	1.00	1.00	0.0	0.0	0.0	0.0	0.0	1.00	1.00
1.00	1.00	1.00	1.00	1.00	0.76	1.00	0.70	1.00	1.00
1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.0	0.0
1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.71	1.00	1.00
1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.71	1.00	1.00
1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.0	1.00

Table 6.4

Observed and Predicted Requirement
Scores in One Task
(Continued)

1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.00	1.00	1.00	1.00	0.0	1.00	1.00	1.00	0.0	1.00
0.0	0.0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.0	1.00	1.00	1.00	1.00	0.0	0.0	0.0	1.00	1.00
1.00	1.00	1.00	1.00	1.00	0.76	1.00	0.70	1.00	1.00
1.00	1.00	1.00	1.00	0.0	1.00	0.0	1.00	1.00	0.0
1.00	1.00	1.00	0.0	1.00	1.00	0.0	1.00	1.00	1.00
1.00	1.00	1.00	1.00	1.00	0.76	1.00	0.70	1.00	1.00
1.00	1.00	0.92	1.00	1.00	0.75	0.65	0.70	0.88	0.88
1.00	1.00	0.0	1.00	1.00	0.75	1.00	0.70	1.00	1.00
1.00	1.00	1.00	1.00	1.00	1.00	0.0	0.70	1.00	1.00
1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.00	0.0	0.0	1.00	1.00	0.0	0.0	1.00	1.00	1.00
1.00	0.96	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.0
1.00	0.97	0.92	0.93	0.71	0.75	0.65	0.68	0.88	0.86

REFERENCES

[1] DEMPSTER, A. P., N. M. LAIRD and D. B. RUBIN (1977). Maximum likelihood from incomplete data via the EM algorithm. J. Roy Statist. Soc. (B), 39, pp. 1-38.

[2] LITTLE, R. J. A. (1979). Maximum likelihood inference for multiple regression with missing values: a simulation study. J. Roy Statist. Soc. (B), 41, pp. 76-87.

[3] SUNDBERG, R. (1974). Maximum likelihood theory for incomplete data from an exponential family. Scand. J. Statist., 1, pp. 49-58.

[4] WU, C. F. (1983). On the convergence properties of the EM algorithm. Ann. of Statist., 11, pp. 95-103.

[5] ZACKS, S. (1981). Parametric Statistical Inference: Basic Theory and Modern Approaches, Pergamon.

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